$\Gamma\,(0,\,L)$ is found from equation

$$\operatorname{Re}\left[\frac{Bl^3}{3c}+\left(\frac{Bl^6V^2}{9c^2}+\frac{2}{3}\,\frac{Bl^3}{c}\,p+\frac{I}{c}\,p^2\right)^{1/2}+\left(-\frac{m}{EJ}\right)^{1/4}p^{1/2}\right]=0$$

and lies in the right half-plane if $\max_x\,(IV^2\,/\,c)>1$. Thus a rod of sufficient length is stable if $\max_x\,(IV^2\,/\,c)<1$ and unstable if $\max_x\,(IV^2\,/\,c)>1$.

The concepts developed above can also be used in the problems which can be reduced to infinite systems of ordinary differential equations, such as e. g. the problems of hydrodynamic stability.

The author thanks A.G.Kulikovskii for valuable comments and V.Ia.Levchenko for discussing the problems related to the present paper.

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ON THE "EQUIVALENCE" RULE FOR FLOWS OF PERFECT GAS

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An extension of the equivalence of "area" rule [1, 2] is presented. The rule was initially derived for stationary flows of perfect (inviscid and non-heat-conducting) gas past slender fine pointed bodies (or blunted bodies in the hypersonic flow case) whose transverse dimensions are small in comparison with their length. According to that rule the wave drag of a three-dimensional body is equal to the wave drag of an axisymmetric body with the same distribution of cross-sectional areas along the axis. The rule is extended here to stationary and nonstationary flows past nonslender bodies and to internal flows, using the procedure of averaging with respect to the angular variable of a cylindrical system of coordinates. That procedure is, strictly speaking, valid for nearly axisymmetric bodies. However the numerical solutions obtained by the authors for a fairly wide range of external and internal problems show that the generalized equivalence rule is applicable to substantially nonaxisymmetric configurations (*) (see next page).

1. The aim of the present investigation is to prove the applicability of the equivalence rule to nearly axisymmetric stationary and nonstationary flows of perfect gas. Deviation of a flow from the axisymmetric pattern can be caused by the extensiveness of streamlined surfaces and by the lack of symmetry of initial and boundary conditions, as well as by force fields and external sources of energy and mass, whenever these are present in the problem. Let us assume that these factors are such that the parameters of the stream are almost everywhere close to the parameters of some "reference" axisymmetric flow in which these factors are absent. Small neighborhoods of discontinuity surfaces resulting from the distortion of the reference axisymmetric flow and which in what follows will be called (contrary to conventional terminology) strong discontinuities. Substantially nonaxisymmetric discontinuities of low intensity, subsequently called "weak", may also be present in the stream.

Investigation of this kind of flows is conveniently carried out in a cylindrical system of coordinates $xy\varphi$, whose x-axis is oriented in such a way as to ensure a weak dependence of the streamlined surfaces and other conditions on the angle φ.

We denote the projections of the velocity vector $\mathbf{u}$ on the axes of the cylindrical system of coordinates by u, v and w, and write the condition of impenetrability of streamlined surfaces in the form

$$uf_x^{\,w} - v + wf_\varphi^{\,w}/y + f_t^{\,w} = 0 \quad \text{for } y = f^w(t, x, \varphi) \tag{1.1}$$

where $y = f^w(t, x, \varphi)$ is the equation of the streamlined surface, f^w is a known function of its arguments, and the subscripts denote the related partial derivatives.

We introduce the axisymmetric surfaces $y = F^w(t, x)$, which are close to the three-dimensional surfaces and are defined by equations of the form

$$y = F^w(t, x) + \varepsilon_w \delta f^w(t, x, \varphi) \tag{1.2}$$

We define functions F^w and δf^w by equalities

$$F^w(t, x) = \frac{1}{2\pi}\int_0^{2\pi} f^w(t, x, \varphi)\, d\varphi, \quad \varepsilon_w \delta f^w(t, x, \varphi) = f^w(t, x, \varphi) - F^w(t, x) \tag{1.3}$$

where ε_w is the maximum (in modulo for all t, x and φ) value of the remainder appearing in the right-hand part of the second equality. Owing to this $|\delta f^w| \leqslant 1$. For nearly axisymmetric bodies $\varepsilon_w \ll 1$ (it is assumed here and subsequently that all quantities are normalized with respect to characteristic parameters of the considered problem), and F^w, as determined by (1.3), is the same as the mean "over the area" to within ε_w, i.e.

$$F^w(t, x) = \left\{\frac{1}{2\pi}\int_0^{2\pi} [f^w(t, x, \varphi)]^2\, d\varphi\right\}^{1/2} \tag{1.4}$$

Note that the smallness of ε_w in (1.3) is generally insufficient for ensuring the closeness of the stream pattern to axisymmetric, since such smallness does not imply the

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*) The authors became recently aware of M.N.Kogan's paper on the drag of bodies of a shape close to that of bodies of revolution, Inzh. Zh., Vol. 1, № 3, 1961, in which the equivalence rule is extended to sub- and supersonic uniform flows past nonslender bodies of a shape close to that of bodies of revolution.

smallness of the derivative $f_\varphi{}^w$ in the condition of impermeability (1.1) because its order of magnitude is determined not only by $\varepsilon_w \delta f^w$, but also by the pattern of its variation with respect to φ. If, for instance, the variation is such that $\delta f_\varphi{}^w \sim 1 / \varepsilon_w$, then $f_\varphi{}^w \sim 1$. In the case of a considerable twist of the flow, in which $w \sim u$ or $w \sim v$, (1.1) implies a considerable deviation of the flow from axial symmetry (at least in the body neighborhood) even for $\varepsilon_w \ll 1$ in (1.3). Consequently, in the subsequent analysis we assume that for the considered surfaces $\delta f_\varphi{}^w \sim 1$.

By analogy to (1.2) we define surfaces of strong discontinuities $y = f^s(t, x, \varphi)$ by

$$y = F^s(t, x) + \varepsilon_s \delta f^s(t, x, \varphi) \qquad (1.5)$$

where functions F^s and δf^s and parameters ε_s are defined in the same manner as F^w, δf^w and ε_w in (1.2), and in the general case $\varepsilon_s \neq \varepsilon_w$.

Let ω be a three-dimensional flow region. We introduce an axisymmetric region Ω bounded by surfaces $y = F^w(t, x)$ with functions F^w defined by (1.3) or (1.4) and consisting of axisymmetric subregions Ω_i separated by axisymmetric surfaces of strong discontinuities $y = F^s(t, x)$. Everywhere outside the small neighborhoods of boundaries of Ω_i the fields of gas parameters in ω and Ω are assumed to be the same, while in such neighborhoods they are obtained by analytic continuation from ω. Because of this it is possible to assume that throughout Ω_i parameters of the stream satisfy the input equations of the three-dimensional flow, which are valid in ω.

2. Let $\sigma = \sigma(t, x, y, \varphi)$ be an arbitrary parameter of the stream, which in accordance with the described procedure is determined not only in ω, but also in the axisymmetric region Ω. This makes it possible to carry out averaging with respect to φ by formula

$$\Sigma(t, x, y) \equiv \langle \sigma \rangle = \frac{1}{2\pi} \int_0^{2\pi} \sigma(t, x, y, \varphi)\, d\varphi$$

Here and subsequently a capital letter denotes the related quantity and integration with respect to φ is carried out with other variables fixed. We can now write similarly to (1.2) for both ω and Ω

$$\sigma = \Sigma(t, x, y) + \varepsilon_\sigma \delta\sigma(t, x, y, \varphi) \qquad (2.1)$$

where parameter ε_σ is the maximum absolute value of the remainder $(\sigma - \Sigma)$ (throughout the considered region of independent variables) on the strength of which $|\delta\sigma| \leqslant 1$.

In the axisymmetric region Ω the mean parameters, i.e. Σ, may become discontinuous only at transition through the axisymmetric surfaces $y = F^s(t, x)$ defined above, while $\delta\sigma$ may at the same time have stepwise increments also at essentially non-axisymmetric surfaces of weak discontinuities.

The functions which specify the initial distributions of parameters (at $t = 0$), including that of parameters in the oncoming stream (e.g. for $x = -\infty$), the force and other fields associated with external sources can also be represented in the form (2.1). Let $\varepsilon_0, \varepsilon_\gamma, \varepsilon_\alpha, \ldots$ be small parameters corresponding to associated distributions. These parameters are generally independent and may even be of different orders of magnitude. This is even more true with respect to parameters ε_σ which define the deviation from axial symmetry of distributions of stream parameters. Nevertheless, the introduction of $\varepsilon = \max(\varepsilon_w, \varepsilon_0, \varepsilon_\gamma, \varepsilon_\alpha, \varepsilon_s, \varepsilon_\sigma)$ makes it possible to write expressions for any of formulas (1.2), (1.5) and (2.1) as

$$y = F^w(t, x) + \varepsilon \Delta f^w(t, x, \varphi) \tag{2.2}$$
$$y = F^s(t, x) + \varepsilon \Delta f^s(t, x, \varphi)$$
$$\sigma = \Sigma(t, x, y) + \varepsilon \Delta \sigma(t, x, y, \varphi)$$

Expressions for the distribution of parameters in the oncoming stream, as well as for power and other external fields at $t = 0$ are similar. In accordance with the definition ε for any function ζ we have $\Delta\zeta = (\varepsilon_\zeta / \varepsilon)\,\delta\zeta$ and, since $\varepsilon_\zeta \leqslant \varepsilon$, the maximum of $|\Delta\zeta|$ evidently does not exceed unity. Moreover, owing to the method of averaging with respect to φ

$$\int_0^{2\pi} \Delta\zeta\, d\varphi = \frac{\varepsilon_\zeta}{\varepsilon} \int_0^{2\pi} \delta\zeta\, d\varphi = 0 \tag{2.3}$$

Let $\chi(\sigma_1, \sigma_2, \ldots)$ be an arbitrary continuous function of stream parameters σ_1, $\sigma_2, \ldots$. Using expressions (2.2), the expansion of χ in $(\sigma_1 - \Sigma_1), \ldots$ and the property (2.3), it is now possible to show that

$$\langle \chi(\sigma_1, \sigma_2, \ldots) \rangle = \chi(\Sigma_1, \Sigma_2, \ldots) + O(\varepsilon^2) \tag{2.4}$$

This implies in particular that the thermodynamic relationships between averaged parameters are to within ε the same as the relationships between actual quantities.

3. In conformity with the definitions given above we have for any parameter at the surface of the body, e.g. for the pressure $p^w(t, x, \varphi)$

$$p^w(t, x, \varphi) = P(t, x, f^w) + \varepsilon \Delta p(t, x, f^w, \varphi) = \tag{3.1}$$
$$P^w(t, x) + \varepsilon [(\partial P / \partial y) \Delta f^w + \Delta p]_{y=F^w} + O(\varepsilon^2)$$

where $P^w(t, x) = P(t, x, F^w)$. Taking into account this expansion, the first of formulas (2.2), and the property (2.3), it is possible to show that the projection of the x-axis of pressure forces acting at instant of time t on the surface of the body between cross sections $x = x_1$ and $x = x_2$ is

$$X(t, x_1, x_2) = \int_{x_1}^{x_2} \int_0^{2\pi} p^w(t, x, \varphi)\, f^w(t, x, \varphi)\, f_x^{\,w}(t, x, \varphi)\, d\varphi\, dx = \tag{3.2}$$
$$2\pi \int_{x_1}^{x_2} P^w(t, x)\, F^w(t, x)\, F_x^{\,w}(t, x)\, dx + O(\varepsilon^2)$$

It follows from this that X and, as can be readily shown, also the coefficient of wave drag are determined by the integral of the averaged (with respect to φ) parameter taken along the surface of the axisymmetric configuration, with an accuracy to within ε, equivalent to the initial three-dimensional configuration as regards the distribution of the radius (or of the area of cross section within the same accuracy) averaged with respect to x.

It is shown below that the equations, and also the initial and boundary conditions which determine the averaged quantities are equations and conditions of the axisymmetric flow in Ω. Initial three-dimensional configurations, forces and other external sources, and the functions which define the parameter distribution at $t = 0$ and at the boundaries of the considered region (when such are present) are replaced by their axisymmetric equivalents by using the procedure of averaging with respect to φ. That procedure may be considered to be the extension of the equivalence rule to any arbitrary perfect gas flows close to the axisymmetric pattern.

Below we present an outline of the proof of the stated rule.

Any of the equations of flow, i.e. the equation of continuity, the three equations of momentum, and the equation of energy can be represented in the form

$$\frac{\partial ya}{\partial t}+\frac{\partial yb}{\partial x}+\frac{\partial yc}{\partial y}+\frac{\partial e}{\partial \varphi}+g=0 \qquad (3.3)$$

where a, b, c and e are known functions of the stream parameters and of independent variables (e.g. of specified functions of time and coordinates if external forces are present), while g may or may not be present. Equations (3.3) are satisfied in the continuity subregion of parameters of axisymmetric regions Ω_i. Substantially nonaxisymmetric surfaces of weak discontinuities (in the meaning given above) may exist inside Ω_i. If $\varphi = \Phi(t, x, y)$ is the equation of such surface and $[a]$, $[b]$, ... are the remainders of values of parameters a, b,... at the discontinuity, the corresponding law of conservation has the form

$$y(\Phi_t[a]+\Phi_x[b]+\Phi_y[c])-[e]=0 \qquad \text{for } \varphi=\Phi(t\ \ x,\ y) \qquad (3.4)$$

Integrating (3.3) with respect to φ for fixed t, x and y over each interval in $0 \leqslant \varphi \leqslant 2\pi$, of the parameter discontinuity and adding the derived equations with allowance for (3.4), after some transformations, we obtain the equation

$$\frac{\partial y\langle a\rangle}{\partial t}+\frac{\partial y\langle b\rangle}{\partial x}+\frac{\partial y\langle c\rangle}{\partial y}+\langle g\rangle=0 \qquad (3.5)$$

which is valid in every axisymmetric subregion Ω_i. Substituting related functions of averaged parameters for mean $\langle a\rangle$, ... (e.g. RU for $\langle \rho u\rangle$), which we denote here by A,..., we finally obtain the equation

$$\frac{\partial yA}{\partial t}+\frac{\partial yB}{\partial x}+\frac{\partial yC}{\partial y}+G=0 \qquad (3.6)$$

which, unlike the exact equation (3.5), is accurate to within ε.

The relationships at axisymmetric surfaces of strong discontinuities are derived by linearizing related laws of conservation which are satisfied at the initial nonaxisymmetric surfaces and transferring all parameters to axisymmetric surfaces $y = F^s(t, x)$, using formulas of the kind of (3.1),and then integrating with respect to φ from 0 to 2π. The resulting equation which complements (3.6) at discontinuity surfaces with an accuracy to within ε, has the form

$$F_t^s[A]+F_x^s[B]-[C]=0 \qquad \text{for } y=F^s(t,x) \qquad (3.7)$$

Applying the same procedure to the condition of impermeability (1.1), we obtain with the same accuracy

$$UF_x^w-V+F_t^w=0 \qquad \text{for } y=F^w(t,\ x) \qquad (3.8)$$

Initial conditions and the conditions in the oncoming stream (and at boundaries of Ω other than surfaces of bodies, whenever such are present) are obtained for P, U,... by averaging with respect to φ, in accordance with the definition of averaged values, the initial and boundary distributions of p, u,.... These conditions together with Eqs. (3.6), the relationships at strong discontinuities, and the condition of impermeability (3.8) define a certain problem of axisymmetric flow in Ω. The solution of that problem yields the distribution of P^w over the streamlined surface and, in accordance with

(3.2), the integral of pressure forces X.

Since terms containing ε^2 are omitted in the right-hand parts of equations used for solving the problem, one would expect the error of determination of P^w, and consequently also of X, to be of that order. However, in some cases this error may depend not only on the magnitude of neglected terms, but also on dimension D of that part of region Ω in which the flow determines P^w along the considered section of the body. Thus, for instance, in the case of stationary supersonic flows $D \sim 1$ must be additionally specified. If $D \sim 1 / \varepsilon$, the error of determination of P^w may be of the order of ε.

4. The procedure followed above is somewhat similar to that used for averaging with respect to areas which yields equations for one-dimensional flow in channels. The same fairly simple and general procedure can be used for substantiating the application of the equivalence rule to more complex flows (e.g. nonequilibrium flows). Various versions of this procedure are also possible and may prove advantageous in some cases. Let us consider one of such variants which for simplicity's sake we apply to a stream in which there are no markedly nonaxisymmetric surfaces of weak discontinuities (in the meaning defined above).

Let the small parameters ε_k, where $k = 1, \ldots, n$, define the various factors which cause the flow to be three-dimensional, such as: deviation of the body shape from axial symmetry, initial and boundary distributions, and also forces and other external sources. Let us assume that any of the flow parameters can be expanded into series in integral powers of ε_k

$$\sigma = \sigma_0(t, x, y) + \sum_{k=1}^{n} \varepsilon_k \sigma_k(t, x, y, \varphi) + \ldots \tag{4.1}$$

where σ_0 is the solution of some axisymmetric problem and dots denote subsequent terms of the expansion.

To derive the conditions which determine σ_k, we continue analytically the solution of the axisymmetric problem into the neighborhood of the boundaries of regions ω_i and, using formulas of the kind of (3.1) together with the condition of impermeability, transfer the relationships at strong discontinuities to related axisymmetric surfaces. Substituting expansions (4.1) into the obtained conditions and taking into account that functions σ_0 provide the solution of the axisymmetric problem, we obtain in the conventional manner linear equations which relate σ_k to each other, and in the condition of impermeability also to the function which defines the deviation of the shape of the body from axial symmetry. If the equation defining the shape of the body is

$$y = f_0^w(t, x) + \varepsilon_1 f_1^w(t, x, \varphi)$$

the condition of impermeability (1.1) yields n relationships of the kind

$$v_k = u_k f_{0x}^w + (f_{1t}^w + u_0 f_{1x}^w + w_0 f_{1\varphi}^w / f_0^w) \delta_{1k} \quad \text{for} \quad y = f_0^w(t, x) \tag{4.2}$$

where δ_{ik} is the Kronecker delta and $k = 1, \ldots, n$. The linearized equations at strong discontinuities and the initial and boundary conditions different from (4.2) are of a similar form.

By linearizing the nonlinear input system we obtain in a similar manner n linear systems for σ_k which differ only by their right-hand parts. Any of these systems is of the form

$$L(\sigma_k) = g_k \tag{4.3}$$

where $L(z)$ is a linear differential operator acting on the column vector z; σ_k is a column vector whose components are $u_k, v_k, \ldots$, and g_k is a column vector whose components (or one of the components) is nonzero when external forces or sources of mass and energy are nonsymmetric. The coefficients of operator $L(z)$ are expressed in terms of parameters of the axisymmetric solution and its partial derivatives and, consequently, they and the coefficients in (4.2) are independent of φ. It is convenient to include in system (4.3) the finite linear relationships between the perturbations of thermodynamic parameters, which are obtained by the linearization of the equation of state.

If expansions (4.1) are satisfied

$$X(t, x_1, x_2) = X_0(t, x_1, x_2) + 2\pi \int_{x_1}^{x_2} \Big\{ \varepsilon_1 p_0 \left(f_0^{\,w} \langle f_{1x}^{w} \rangle + f_{0x}^{w} \langle f_1^{w} \rangle \right) + f_0^{\,w} f_{0x}^{w} \sum_{k=1}^{?} \varepsilon_{\kappa} \langle p_k \rangle \Big\} dx + \ldots \qquad (4.4)$$

where X_0 is the integral of forces acting on the axisymmetric body, $\langle z \rangle$ is the integrand averaged with respect to φ for $y = f_0^{w}$, and dots denote higher order terms.

The equations and conditions which for $y = f_0^{w}$ determine $\langle \sigma_k \rangle$ and, consequently, also $\langle p_k \rangle$ which appear in (4.4), are obtained by integrating system (4.3), conditions (4.4), and other linearized initial and boundary conditions with respect to φ from 0 to 2π. Taking advantage of the arbitrariness of the selection of the axisymmetric surface $y = f_0^{w}$ and of the axial symmetry of the distribution of force, initial parameters, etc., and setting $f_0^{w} = F^{w}$, we determine the remaining indicated functions by averaging with respect to φ and find that $\langle f_1^{w} \rangle$, $\langle f_{1x}^{w} \rangle$, $\langle g_k \rangle$, ... vanish. Moreover, since in the absence of weak discontinuities $\langle z_\varphi \rangle = 0$, the averaged equations and conditions do not contain derivatives averaged with respect to φ. As the result the system of equations and conditions which determines the mean values of all additions becomes not only linear, but also homogeneous. If the conditions specified at the end of Sect. 3 are satisfied, the trivial solution which satisfies all equations and conditions of the derived linear problem yields the coefficients of the first terms of expansion (4.1) with an error of the order of $O(\varepsilon_k)$ In such cases $X(t, x_1, x_2) = X_0(t, x_1, x_2)$ with an accuracy to within $O(\varepsilon_k)$.

The described method was used in [3, 4] for deriving equations which determine certain integral properties of slender three-dimensional configurations (bodies with through-flow channels and wings) in a supersonic stream (*). It differs from that described in Sects. 1–3 in that the averaging with respect to φ is preceded by the linearization of not only of the condition of impermeability and of relationships at strong discontinuities, but also of the differential equations of flow. Superficially the two methods appear to be equivalent. Although for many kinds of flows this is true, it is not necessarily so in the case of other flows. In a number of important cases (e.g. transonic speeds) expansions of the kind (4.1) with coefficients determined by system (4.3) – on whose linearity the preceding analysis is based – are not valid. In such cases the related system is nonlinear, and cannot be analyzed by the very simple method described here. However, in the initial approach to the prblem only the smallness of parameter deviation from some averaged (with respect to φ) values is important and independent of the kind of equations

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*) M. N. Kogan (see footnote on P. 952) had proved by the same method the generalized equivalence rule for nonslender bodies of a shape close to that of bodies of revolution.

which define the deviations. It should be stressed that only when expansions (4.1) are valid, is the order of magnitude of parameter ε introduced in (2.2) the same as that of $\max \varepsilon_k$.

5. The following examples illustrate the applicability of the equivalence rule to stationary flows. They include external and internal problems of transonic and supersonic flows of perfect gas with the specific heat ratio $\varkappa = 1.4$.

The first example is that of flow past cones of various cross-sectional form. Calculations were carried out with the use of the process, described in [5], of establishing a system of spherical coordinates in terms of the radial variable and integrating the equations of three-dimensional flow, using the difference method proposed in [6, 7].

All of the investigated examples dealt with flows past bodies at zero angle of attack. Besides cones of circular and elliptical cross sections, cones with cross sections close to a square and to an equilateral triangle were considered. The latter (as well as nozzles of similar cross-sectional shape) are called below "square" and "triangular", respectively. The boundary of a triangular cone cross section consisted of three straight lines joined by circular arcs. The radius of the joining circle was $l^0x / 2\sqrt{3}$, where l^0x is the length of the straight line segment for any x. Cross sections of square cones were similarly constructed with the straight line segment length and the joining radius equal, respectively, l^0x and $l^0x / 2$. Coefficients l^0 were obtained from the condition of equality (for equal x) of cross-sectional areas of the investigated cones to corresponding areas of a circular cone with the vertex half-angle θ_k. Applying this condition to elliptical cones with a fixed ratio of semiaxes a / b of the generating ellipse, we determined the quantity $a^0 = a / x$.

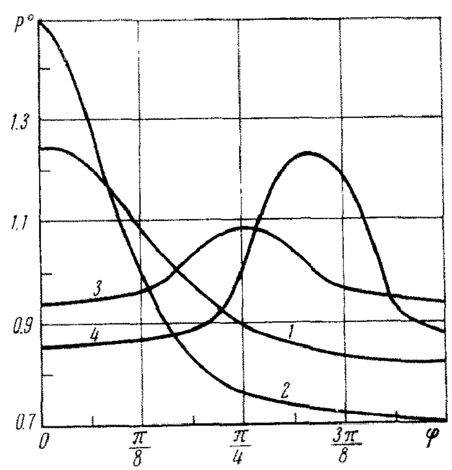

Fig. 1

The results of calculations of the supersonic stream at $M_\infty = 5$ flowing past square, triangular, and elliptic cones with cross-sectional areas equivalent to those of a circular cone with $\theta_k = 20°$ are shown in Fig. 1, where $p°$ is the ratio of surface pressure on the investigated cone to that at the corresponding part of the circular cone. Curves 1 – 4 relate to two elliptic cones with $a / b =$ 1.5 and 2.0, and to a square and a triangular cone. When examining Fig. 1 it is necessary to bear in mind that there are two, four and three planes of symmetry for the elliptic, square, and triangular cones, respectively, and that the maxima of $p°$ lie at the points of the contour farthermost from the x-axis, i.e. at the intersection line of the planes of symmetry. It will be seen from this figure that the deviation of $p°$ from unity is in this case considerable. However the values obtained by averaging the considered distributions with respect to φ, which in accordance with (3.1) coincide with the values of $p°$, were for the same cases equal to 0.96, 0.87, 1.01, and 0.93. The considerably smaller deviation of $p°$ from unity confirms the general derivation of the extented equivalence rule by which the difference between the averaged and the axisymmetric parameters is $O(\varepsilon^2)$ when the deviation of local from averaged parameters is of the order of ε.

The coefficients of wave drag were calculated for the above cones by formula

$$c_x = 2(X - p_\infty S) / S\rho_\infty q_\infty^2$$

where ρ is the density, q is the velocity modulus, the subscript ∞ denotes parameters of the oncoming stream, and S is the area of a cross section. For conical bodies X and S are linear functions of x^2, and c_x is independent of x. For the considered conical bodies c_x was found to be equal to 0.260, 0.256, 0.255 and 0.251, while for the equivalent (with respect to area) circular cone it is 0.261, which corresponds to a maximum deviation of 4%.

Under conditions different from those considered above (as regards M_∞ and θ_k) the concept of error of the equivalence rule yields for an elliptic cone with $a / b = 2.0$ the following values. For $M_\infty = 5$ the coefficients of wave drag of three elliptic cones equivalent to circular cones with $\theta_k = 10$, 20, and 25° are: 0.076, 0.26 and 0.38, respectively. For circular cones $c_x = 0.075$, 0.26 and 0.37, which corresponds to a maximum devaition not exceeding 3%. For an elliptic cone equivalent to a circular one with $\theta_k = 20°$, $c_x = 0.29$, 0.26, 0.24, respectively, for $M_\infty = 3.0$, 5.0 and 10. For a curcular cone under the same conditions $c_x = 0.28$, 0.26 and 0.25, hence in this case the deviation is of the same order of magnitude.

The flow at $M_\infty = 5$ past four three-dimensional pointed bodies was also calculated. The cross sections of each of these bodies were of fixed configuration consisting of a conical nose equivalent to a circular cone with $\theta_k = 20°$ and joined to a tangent parabolic ogive. All cones considered above (for $M_\infty = 5$ and $\theta_k = 20°$) were used as nose cones. The x- and y-coordinates and the length of bodies were normalized with respect to the length of the nose, and the total length of the body expressed in these units was equal ten. The investigated configurations were equivalent to bodies of revolution whose parabolic generatrix was defined by the equation $y = \sqrt{\alpha + \beta x}$ tangential to the nose cone. This configuration resulted in a twenty-fold increase of the cross-sectional area between $x = 1$ and $x = 10$.

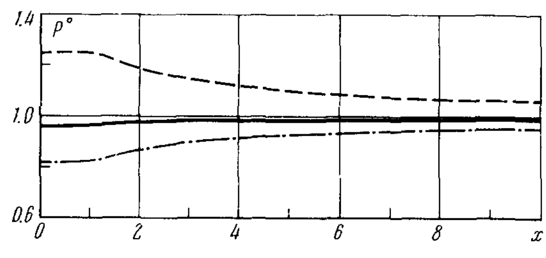

Fig. 2

The nonuniformity of pressure distribution over the surface of a body of elliptic cross section ($a / b = 2.0$) (in this case the nonuniformity is maximum) is shown in Fig. 2, where the dash and the dash-dot lines relate to the distribution of $p°$ along generatrices lying in planes of symmetry which pass through the semimajor and semiminor axes of cross section. The continuous curve shows the distribution of the related parameter averaged with respect to φ. The coefficients of wave drag c_x of the considered bodies of elliptic ($a / b = 1.5$ and 2.0), square, and triangular cross sections were equal to the c_x of equivalent bodies of revolution within the computation accuracy (1 – 2 %).

The following examples deal with the flow in the supersonic parts of three three-dimensional nozzles equivalent to an axisymmetric nozzle whose generatrix consisted of an arc of circle of radius $r = 2$ with its center on the y-axis and tangent to a section of parabola $y = \sqrt{\alpha + \beta x}$. All dimensions were normalized with respect to radius r_0 of the initial cross section. The coefficients α and β in this formula were such as to ensure the tangency of the circle and the parabola and a four-fold expansion of investigated nozzles in y from $x = 0$ to $x = 10$, i.e. $y(10) = 4$. The cross sections of the three-dimensional nozzles were of the same shape as that of the elliptic ($a / b = 2.0$), square, and triangular cones considered above. At the inlet cross sections (at $x = 0$) of all nozzles the stream was supersonic and uniform at $u_0 = 1.1$ normalized with respect to the critical velocity q_*. Computations were carried out in elliptic coordinates and the difference method [6, 7] was used.

Pressure distribution at the wall along lines of its intersection with planes of symmetry is shown in Fig. 3 constructed on the same basis as in Fig. 2. The dash and the dash-dot

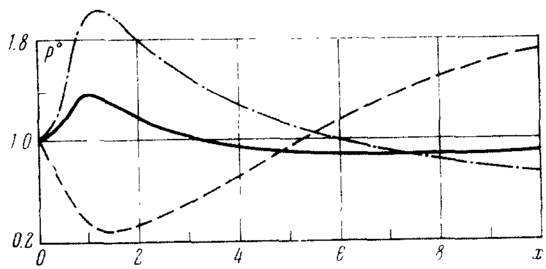

Fig. 3

lines relate to points of the major and minor axes of the cross section, respectively, and the solid line represents the pressure at the wall averaged with respect to φ. As previously, the pressure was normalized with respect to the pressure at the wall of the equivalent axisymmetric nozzle (at the same x).

It will be seen from Fig. 3 that in this example the nonuniformity of pressure distribution with respect to φ is considerably greater than in the external problem. The same can be said about the ratio of averaged parameters to their corresponding values in the axisymmetric case, although it is considerably closer to unity. The equivalence rule yields even better results for the integrals of pressure forces, which for $x_1 = 0$ and $x_2 = 10$, i.e. for nozzles with an area expansion ratio of 16, were 1.618, 1.919 and 1.616 for elliptic, square, and triangular nozzles, respectively, while for the axisymmetric nozzle $X = 1.634$ (X and the stream momentum are normalized with respect to $\rho_* q_*^2 y_0^2$, where ρ_* is the critical density of the stream). The momentum at outlet cross sections of these nozzles were, respectively, 6.964, 6.965, 6.962 and 6.980, which corresponds to a maximum error of 0.25%. The error of determination of momentum at all cross sections between $x = 0$ and $x = 10$ is of the same order of magnitude (not more than 0.5 %).

Two examples of transonic flow in a nozzle and past the afterbody of a semi-infinite cylinder of elliptic cross section ($a / b = 2.0$) were also computed. In both these cases the stationary distribution of paramters were obtained in the course of determination with respect to time, as described in [8 – 10].

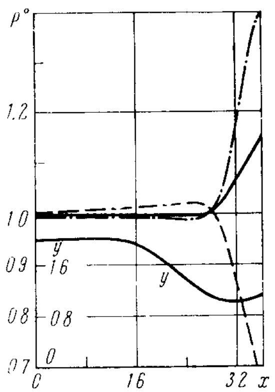

Fig. 4

The three-dimensional Laval nozzle was equivalent as to its area to an axisymmetric nozzle whose meridian cross section is shown in the lower part of Fig. 4. In the upper part of that figure, which is similar to Fig. 3, are shown curves of pressure distribution at two generatrices of the wall, and also the variation of related averaged quantity. The difference between the momenta of the three-dimensional and the axisymmetric nozzles (of the order of 0.4%) is within the computation accuracy. A similar situation was revealed by the computation of flow at $M_\infty = 0.9$ past the afterbody equivalent to that considered in [10], which at $x = 0$ was smoothly joined to a semi-infinite cylinder whose radius was taken as the characteristic dimension. The afterbody generated by a circle of radius $r = 9.68$ was joined at point $x = 2.5$ to another semi-infinite cylinder ($y \equiv 0.67$).

The above computations prove the validity of the generalized equivalence rule far beyond the restrictions imposed in its derivation. We note that an experimental proof of the validity of the area rule for hypersonic flow past blunted slender bodies was given in [11].

In conclusion the authors thank A.B. Vatazhin and G.G. Chernyi for valuable discussions.

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THEORY OF THE HYPERSONIC VISCOUS SHOCK LAYER AT HIGH REYNOLDS NUMBERS AND INTENSIVE INJECTION OF FOREIGN GASES

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The hypersonic flow around smooth blunted bodies in the presence of intensive injection from the surface of these is considered. Using the method of external and internal expansions the asymptotics of the Navier-Stokes equations is constructed for high Reynolds numbers determined by parameters of the oncoming stream and of the injected gas. The flow in the shock layer falls into three characteristic regions. In regions adjacent to the body surface and the shock wave the effects associated with molecular transport are insignificant, while in the intermediate region they predominate. In the derivation of solution in the first two regions the surface of contact discontinuity is substituted for the region of molecular transport (external problem). An analytic solution of the external problem is obtained for small values of parameters $\varepsilon_1 = \rho_\infty / \rho_s^*$ and $\delta = \rho_w^{-1/2} v_w^* / \rho_\infty^{1/2} v_\infty$ in the form of corresponding series expansions in these parameters. Asymptotic formulas are presented for velocity profiles, temperatures, and constituent concentration across the shock layer and, also, the shape of the contact discontinuity and of shock wave separation. The derived solution is compared with numerical solutions obtained by other authors. The flow in the region of molecular transport is defined by equations of the boundary layer with asymptotic conditions at plus and minus infinity, determined by the external solution (internal problem). A numerical solution of the internal problem is obtained taking into consideration multicomponent diffusion and heat exchange. The problem of multicomponent gas flow in the shock layer close to the stagnation line was previously considered in [1] with the use of simplified Navier-Stokes equations.

The supersonic flow of a homogeneous inviscid and non-heat-conducting gas around blunted bodies in the presence of subsonic injection was considered in [2 – 7] using Euler's equations. An analytic solution, based on the classic solution obtained by Hill for a spherical vortex, was derived in [2] for a sphere on the assumption of constant but different densities in the layers between the shock wave and the contact discontinuity and between the latter and the body. Certain results of a numerical solution of the problem of intensive injection at the surface of axisymmetric bodies of various forms, obtained by Godunov's method [3], are presented. Telenin's method was used in [4] for numerical investigation